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A Comparative Study of M-APOS, Knisley, and Discovery Learning on Students' Advanced Mathematical Thinking Skills

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Abstract: Advanced Mathematical Thinking (AMT), which includes representation, abstraction, creative thinking, and mathematical proof, is crucial for studying abstract algebra. However, there is limited research specifically investigating how the M-APOS, Knisley, and Discovery Learning models influence students' AMT abilities, especially in the context of self-regulated learning. This study aims to analyze the impact of these three learning models and self-regulated learning on students' AMT abilities, as well as to examine their interactions. This quantitative study utilized a 3×2 quasi-experimental factorial design. The participants were 66 students from the Mathematics Study Program at Universitas Pamulang, selected via purposive sampling. Twenty-two students were assigned to each learning model group. AMT proficiency was evaluated using a four-item essay test validated by experts. Self-regulated learning was assessed using questionnaires and classified as high or low based on the total score. Data were analyzed using two-way ANOVA. The results indicated that the learning model significantly affects AMT ability ($F=9.66$; $p=0.001<0.05$), whereas self-regulated learning does not have a significant effect ($F=1.47$; $p=0.235>0.05$). Furthermore, a significant interaction was found between the learning models and self-regulated learning on AMT ability ($F=3.66$; $p=0.018<0.05$). These findings suggest that the M-APOS model is more effective in enhancing students' AMT and that the learning model's effectiveness is influenced by levels of self-regulated learning.

Abstrak: *Advanced Mathematical Thinking (AMT) yang mencakup kemampuan representasi, abstraksi, berpikir kreatif, dan pembuktian matematis merupakan kompetensi penting dalam pembelajaran aljabar abstrak. Namun, penelitian yang secara khusus menguji pengaruh model pembelajaran M-APOS, Knisley, dan Discovery Learning terhadap kemampuan AMT mahasiswa dengan mempertimbangkan kemandirian belajar masih terbatas. Penelitian ini bertujuan menganalisis pengaruh tiga model pembelajaran dan kemandirian belajar terhadap kemampuan AMT mahasiswa, serta menguji interaksi keduanya. Penelitian ini menggunakan pendekatan kuantitatif dengan desain kuasi-eksperimen faktorial 3×2 pada 66 mahasiswa Program Studi Matematika Universitas Pamulang yang dipilih melalui purposive sampling, dengan masing-masing 22 mahasiswa pada setiap kelompok model pembelajaran. Kemampuan AMT diukur menggunakan tes esai empat butir yang telah divalidasi ahli, sedangkan kemandirian belajar diukur menggunakan angket dan dikategorikan menjadi tinggi dan rendah berdasarkan skor total. Data dianalisis menggunakan ANAVA dua jalur. Hasil penelitian* *menunjukkan bahwa model pembelajaran berpengaruh signifikan terhadap kemampuan AMT* ($F=9,66;p=0,001<0,05$)*, sedangkan kemandirian belajar tidak berpengaruh signifikan* ($F=1,47;p=0,235>0,05$)*. Selain itu, terdapat interaksi yang signifikan antara model pembelajaran dan kemandirian belajar terhadap kemampuan AMT* ($F=3,66;p=0,018<0,05$)*. Temuan ini menunjukkan bahwa model M-APOS lebih efektif dalam meningkatkan AMT mahasiswa dan bahwa efektivitas model pembelajaran dipengaruhi oleh tingkat kemandirian belajar.*

keywords: Advanced mathematical thinking; self-regulated learning; mathematics education; m-apos; quasi-experimental study

INTRODUCTION

Advanced Mathematical Thinking (AMT) is a crucial skill that higher education should prioritize because it significantly aids students in solving complex, proof-based mathematical problems (Wahyuni, 2020; Aristika et al., 2021). AMT generally encompasses four key aspects: representation, abstraction, mathematical creative thinking, and mathematical proof (Wahyuni, 2020; Putri & Awalludin, 2024; Rifa, 2024). These integrated mathematical skills are vital as they enable students to tackle non-routine problems and engage in high-level reasoning (NCTM, 2000). Conversely, students lacking proficiency in AMT often struggle in courses demanding advanced thinking and proof, such as abstract algebra (Saragih et al., 2025; Álvarez et al., 2022).

Abstract algebra explores the abstraction of sets and binary operations, focusing on algebraic structures like groups, rings, and fields (Saragih et al., 2025; Álvarez et al., 2022). The curriculum heavily features discussions of axioms, theorems, and proofs, necessitating analytical, logical, and systematic thinking to resolve abstract problems (Hendriana et al., 2019; Arnawa et al., 2019). Students with strong AMT skills—particularly in representation, abstraction, creative thinking, and proof—will find it easier to grasp and construct arguments within these abstract topics (Wahyuni, 2020; Herlina, 2015a).

In the context of learning mathematics, the ability to represent is considered one of the most essential elements (Absorin & Sugiman, 2018; Herlina, 2015b). Representation involves not only the use of notation, symbols, and formal syntax but also the process of organizing concrete experiences into more abstract concepts and mapping one structure to another (Tall, 2005; Herlina, 2015a). The principles and standards of mathematical learning assert that mathematical representation should be crucial in helping learners understand, communicate, and connect mathematical ideas (National Council of Teachers of Mathematics [NCTM], 2000). Therefore, representational ability is viewed as a core component in the development of AMT.

The second aspect of AMT is the ability of abstraction, which is the capacity to understand mathematical concepts along with their rules and symbols (Herlina, 2015b; Andani et al., 2021). A student demonstrates abstraction ability if they can identify mathematical objects through direct experience, generalize, describe concepts using mathematical language and symbols, and apply these concepts in various contexts (Datreni, 2022; Lestari et al., 2021). The third aspect is the ability to think creatively mathematically. Creative thinking in mathematics is understood as the process of generating new ideas by reconstructing and reorganizing existing knowledge, enabling students to explore various alternative strategies for problem-solving (Putri & Awalludin, 2024; Husnidar & Hayati, 2022).

The fourth aspect is the ability to construct mathematical proofs, which involves understanding the problem, formulating logical arguments, and compiling valid evidence to justify a solution (Arnawa et al., 2019; Rifa, 2024). This ability is closely related to using new methods or strategies to solve problems and verifying the correctness of mathematical statements (Armiati et al., 2019). Mathematical proof plays a crucial role in building a deep conceptual understanding and is often used as an indicator of student success in mastering advanced mathematics topics (Álvarez et al., 2022; Arnawa et al., 2019). Mastery of the four aspects of AMT is determined not only by the strategies and learning models used by instructors but also by internal student factors, such as motivation, initiative to seek information, and self-regulated learning (Sukmawati, 2021; Rifky, 2020). Self-regulated learning significantly influences an individual's success in the learning process (Sukmawati, 2021; Rifky, 2020; Rudi et al., 2020). It is demonstrated by learners' ability to set goals, monitor and evaluate their progress, manage time effectively, persist in tasks, and employ appropriate strategies to solve mathematical problems (Sukmawati, 2021; Rifky, 2020). Self-regulated learners are also responsible, value time, show initiative and autonomy, and can identify and address their learning needs without excessive dependence on others (Rudi et al., 2020; Andani et al., 2021). Thus, self-regulated learning is crucial for understanding mathematical concepts and achieving better learning outcomes, including the mastery of advanced mathematical abilities (AMT) (Datreni, 2022; Lestari et al., 2021).

Current mathematics education emphasizes that students discover and construct mathematical concepts through self-regulated learning, systematic thinking, and the integration of various mathematical abilities (Andani et al., 2021; Husnidar & Hayati, 2021). However, unengaging and monotonous learning environments often reduce student motivation and weaken their capacity for effective self-regulation (Andani et al., 2021; Datreni, 2022; Husnidar & Hayati, 2021; Lestari et al., 2021). A primary cause of this monotony is the limited variety of learning models instructors employ. Therefore, instructors must be creative in selecting and implementing learning models that stimulate AMT while simultaneously fostering students' self-regulated learning (Yerizon et al., 2024; Dewi et al., 2020; Mulyono et al., 2020).

Various studies have shown that several innovative learning models can enhance advanced math skills, including the Modified APOS (M-APOS) model, the Knisley math learning model, and the Discovery Learning model (Yerizon et al., 2024; Anam et al., 2019; Dewi et al., 2020; Mulyono et al., 2020; Izabella & Darsimah, 2021). The M-APOS Model, a modification of the APOS theory, replaces the computer laboratory phase with structured assignments or student worksheets. It maintains the action-process-object-scheme cycle, rooted in the philosophy of Constructivism (Anam et al., 2019; Yerizon et al., 2024; Prasetyo et al., 2021; Vidakovic et al., 2018). These modifications are particularly useful when laboratory facilities are limited or when technical errors from computer programs impede learning objectives (Marsitin & Sesanti, 2018; Mukavhi et al., 2021). Previous research indicates that APOS or M-APOS-based learning can improve students' reasoning and proof skills in topics such as algebra and limits (Marsitin & Sesanti, 2018; Şefik et al., 2021; Priss, 2018; Nisa et al., 2020; Arnawa et al., 2019).

Knisley's mathematical learning Model posits that learning is an experiential process influenced by two primary dimensions: the choice between concrete and abstract experiences, and between reflective observation and active experimentation (Dewi et al., 2020; Mulyono et al., 2020). Based on these dimensions, Knisley proposed four stages of mathematics learning: allegorical, integration, analysis, and synthesis (Knisley, 2003). In this model, instructors gradually transition from storytellers to motivators, resource persons, and trainers, guiding students through these four stages (Mulyono et al., 2020). Empirical findings suggest that Knisley's model can improve students' mathematical communication and creative thinking skills (Dewi et al., 2020; Mulyono et al., 2020).

The Discovery Learning model is a learning approach that fosters active learning through students' self-discovery processes and individual investigation (Izabella & Darsimah, 2021). This model requires students to investigate, explore problems, and maximize their knowledge and skills to construct new concepts. Numerous studies have concluded that Discovery Learning can improve cognitive learning outcomes and enhance students' understanding of mathematical concepts across various educational levels (Sukmawati, 2021; Andani et al., 2021; Datreni, 2022; Husnidar & Hayati, 2021; Lestari et al., 2021; Izabella & Darsimah, 2021).

Although previous studies have examined the effectiveness of the M-APOS, Knisley, and Discovery Learning models, most have investigated each model separately and focused on different outcomes, such as mathematical reasoning, communication, or creative thinking skills (Yerizon et al., 2024; Dewi et al., 2020; Mulyono et al., 2020; Arnawa et al., 2019). Moreover, studies examining Advanced Mathematical Thinking (AMT) in Abstract Algebra courses have generally emphasized instructional approaches without considering important learner characteristics. Research that specifically compares the effects of the M-APOS, Knisley, and Discovery Learning models on students' AMT abilities in Abstract Algebra, while considering self-regulated learning as a moderating variable, remains limited (Prasetyo et al., 2021; Marsitin & Sesanti, 2018; Rifa, 2024). This limitation reveals a research gap concerning how different learning models and levels of self-regulated learning jointly influence students' Advanced Mathematical Thinking abilities.

Addressing this gap constitutes the novelty of the present study. Unlike previous research that examined these learning models separately or focused on different mathematical outcomes, this study simultaneously compares the effects of the M-APOS, Knisley, and Discovery Learning models on students' AMT abilities in Abstract Algebra and investigates the moderating role of self-regulated learning. Therefore, this study provides a more comprehensive understanding of the relationship between instructional models, learner characteristics, and Advanced Mathematical Thinking development in higher mathematics education.

Based on these gaps, this study investigates the effects of the M-APOS, Knisley, and Discovery Learning models on students' Advanced Mathematical Thinking (AMT) abilities in Abstract Algebra courses, while also considering their level of self-regulated learning. Specifically, the objectives are to: (1) compare the AMT abilities of students taught using the M-APOS, Knisley, and Discovery Learning models; (2) compare the AMT abilities of students with high and low levels of self-regulated learning; and (3) analyze the interaction effect between learning models and self-regulated learning on students' AMT abilities (Yerizon et al., 2024; Dewi et al., 2020; Rifa, 2024). In line with these objectives, the study hypotheses are formulated as follows: H_1: There is a significant difference in AMT ability among students taught using the M-APOS, Knisley, and Discovery Learning models. H_2: Students with high levels of self-regulated learning demonstrate significantly higher AMT abilities than students with low levels of self-regulated learning. H_3: There is a significant interaction effect between the learning models (M-APOS, Knisley, and Discovery Learning) and self-regulated learning on students' AMT abilities.

METHOD

This quasi-experimental study employs a pretest-posttest nonequivalent group design to investigate the impact of various learning models and student self-regulated learning on students' Abstract Mathematical Thinking (AMT) abilities. The independent variables are the M-APOS, Knisley, and Discovery Learning models, as well as student self-regulated learning. The dependent variable is students' AMT ability.

The study subjects were students enrolled in the algebra structure course within the Mathematics and Natural Sciences Program at Universitas Pamulang during the research semester. A purposive sampling technique was used to select three classes taught by the same lecturer, utilizing consistent learning achievements, and from the same academic semester level, thereby ensuring comparable academic characteristics.

The study included 66 student respondents, with 22 students in each of the three classes. This allocation was based on the number of active students who met the sample criteria and could participate in the entire research treatment series. The first class received treatment using the M-APOS model, the second class the Knisley model, and the third class the Discovery Learning model. Participant demographic data collected included age, gender, semester, and previous academic achievement in mathematics. Sampled students were approximately 19-22 years old, were from the same academic semester, and had completed relevant prerequisite courses.

Research instruments comprised both test and non-test components. Test instruments measured students' AMT ability, while non-test instruments, in the form of questionnaires, assessed student self-regulated learning. Indicators of self-regulated learning included: (1) confidence, (2) discipline in learning, (3) independence from others, (4) responsibility, and (5) initiative in learning. Based on their questionnaire scores, students were subsequently categorized into groups of high and low self-regulated learning according to established grouping criteria.

The AMT ability test consists of four descriptive questions based on indicators of advanced mathematical thinking skills: representation, abstraction, mathematical creative thinking, and mathematical proof. The problems focus on algebraic structure topics, such as identifying binary operation properties, determining and representing group structures, generalizing from concrete examples to formal definitions, and constructing simple proofs related to group or subgroup properties.

The AMT test is assessed using an analytical rubric to ensure a more objective and structured scoring process. Each problem is evaluated based on several aspects: the accuracy of mathematical representation, the ability to perform abstraction, the novelty or flexibility of resolution strategies, and the accuracy and logic of the proof. Each item is scored on a scale of 0 to 4, where: a score of 0 indicates no answer or an irrelevant answer; a score of 1 indicates a very limited and imprecise answer; a score of 2 indicates a partially correct answer with fundamental errors; a score of 3 indicates a generally correct but incomplete answer; and a score of 4 indicates a complete, logical, and mathematically correct answer. The total AMT score is the sum of the scores for all questions.

Before being used in research, test instruments and questionnaires undergo a two-step validation process by subject matter experts. First, a validator assesses the instrument's content suitability, editorial clarity, language appropriateness, and the relevance of its questions to the research objectives. The validation results guide revisions until the instrument is deemed fit for use. Second, the instrument is empirically tested for validity and reliability. The item validity test determines the extent to which questionnaire items measure the intended construct, while the reliability test ensures consistency of measurement results. This rigorous process of expert content validation and reliability testing ensures the instrument's suitability before its application to the research sample.

Data analysis in this study involves several stages. First, the validity and reliability of both test and non-test instruments are assessed. Second, prerequisite analyses, such as normality and homogeneity tests, are conducted to ensure the data meet the assumptions for parametric analysis. Third, the data are analyzed using a two-way ANOVA to determine: (1) differences in AMT capabilities based on learning models, (2) differences in AMT capabilities based on the level of self-regulated learning, and (3) the interactive effect between learning models and self-regulated learning on students' AMT capabilities. Fourth, if the ANOVA results indicate a significant difference, a post-hoc Tukey test is performed to identify which specific pairs of groups are significantly different.

RESULTS AND DISCUSSION

1.1. Results

There are three classes in this study, namely the first experimental class using the M-APOS learning model, the second experimental class using the Knisley learning model and the control class using the Discovery Learning model. Each class consisted of 22 students. The results of descriptive statistics of experimental and control classes are as follows:

Table 1. Descriptive statistics of experimental class and control class

Description	M-APOS	Knisley	*Discovery Learning*
Average	69,9	60,9	59,2
Median	68,8	59,4	56,3
Standard Deviation	7,0	8,8	10,9
Maximum Value	81,3	75,0	75,0
Minimum Value	56,3	50,0	37,5

Based on the data in Table 1, it can be seen that the average score using the M-APOS model is higher than the other classes (Knisley and Discovery Learning), and while the average of the class using the Knisley model is higher than the average using the Discovery Learning model. The score comparison is clearly visible with

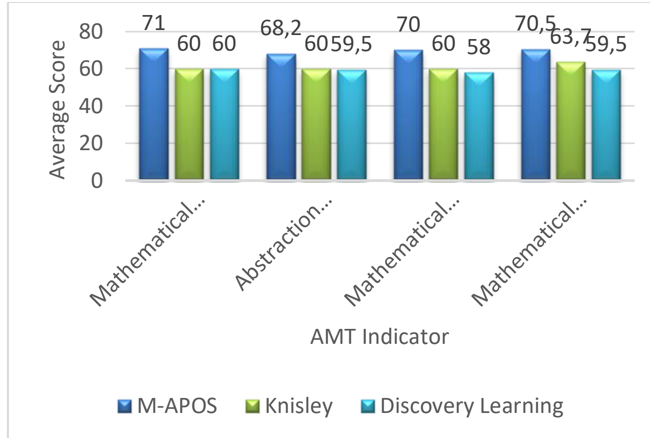

Figure 1: Comparison diagram of average AMT ability

It can be seen that the M-APOS model's class comparison is higher than that of the other classes, namely Knisley and Discovery Learning. To determine whether these differences are statistically significant, a Tukey test was conducted. Before conducting this test, the prerequisite assumptions had to be satisfied.

The prerequisite tests consisted of the normality test and the homogeneity test. The normality test was carried out using the Liliefors test, with the criterion that H_0 is rejected if $L_{count} < L_{table}$, and otherwise accepted. The results showed that all experimental and control class groups were normally distributed. The homogeneity test was conducted on the treatment groups based on the learning model and self-regulated learning. The Bartlett test was used, with the criterion that if the p-value < 0.05, the data are not homogeneous, and vice versa. The Bartlett test result showed a p-value of 0.051, which is greater than 0.05, indicating that the data were homogeneous.

Furthermore, a two-way ANOVA test was conducted to determine whether the learning model and self-regulated learning affected AMT ability. The proposed hypothesis was: "Students' learning outcomes in the form of Advanced Mathematical Thinking (AMT) abilities taught using the M-APOS learning model are higher than those of students taught using the Knisley model and Discovery

Learning model." The results of the two-way ANOVA analysis are presented in the following table.

ANOVA

Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
self-regulated learning	108,51	1	108,51	1,47	0,235	4,17
Learning Model	1425,78	2	712,89	9,66	0,001	3,32
Interaction	392,80	2	196,40	3,66	0,018	3,32
Eror	2213,54	30	4078			
Total	4140,63	35				

Based on the results of the two-lane ANOVA in Table 2, the learning model significantly affects AMT ability ($F = 9.66$, p-value = $0.001 < 0.05$). This indicates significant differences in AMT ability among students taught using the M-APOS, Knisley, and Discovery Learning models. The self-regulated learning factor (high and low) does not show a significant difference in AMT ability ($F = 1.47$, p-value = 0. 235> 0.05). However, there is a significant interaction effect between learning models and self-regulated learning on AMT ability ($F = 3.66$, p-value = 0.018 <0.05).

Table 3. Summary of Tukey Test Results Based on the Learning Model
(Without Distinction of Self-Regulated Learning)

Description	Average	Sign	$R_{Comparison}$	Description
$\lvert A_1-A_2\rvert$	12,50	> R	12,37	Different
$\lvert A_1-A_3\rvert$	14,06	> R		Different
$\lvert A_2-A_3\rvert$	1,56	< R		Same

Description:

A_1 = M-APOS model; A_2 = Knisley model; A_3 = Discovery Learning model. R = Tukey critical value at 5% significance level.

Based on Table 3, the mean difference between the M-APOS and Knisley models ($|A_1-A_2| = 12.50$) and between the M-APOS and Discovery Learning models ($|A_1-A_3| = 14.06$) is greater than the Tukey critical value ($R = 12.37$). This indicates a significant difference in AMT ability between students who studied with the M-APOS model and those who studied with the Knisley and Discovery Learning models. In contrast, the mean difference between the Knisley and Discovery Learning models ($|A_2-A_3| = 1.56$) is smaller than R, indicating no significant difference between these two models

Table 4. Summary of Tukey Test Results Based on a Combination of Learning and Self Regulated Learning Models

Description	Average	Sign	$R_{Comparison}$	Description
$\lvert A_1B_1-A_2B_1\rvert$	14,58	> R		Different
$\lvert A_1B_1-A_3B_1\rvert$	21,88	> R		Different
$\lvert A_2B_1-A_3B_1\rvert$	7,29	< R	12,37	Same
$\lvert A_1B_2-A_2B_2\rvert$	10,42	< R		Same
$\lvert A_1B_2-A_3B_2\rvert$	6,25	< R		Same
$\lvert A_2B_2-A_3B_2\rvert$	4,17	< R		Same

As shown in Table 4, in the group of students with high Summary of Tukey Test Results Based on a Combination of Learning and Self-Regulated Learning Models (B_1), the mean difference between M-APOS and Knisley ($|A_1B_1 - A_2B_1| = 14.58$) and between M-APOS and Discovery Learning ($|A_1B_1 - A_3B_1| = 21.88$) was greater than $R = 12.37$, indicating a significant difference. However, other comparisons, in both the high and low self-reliance groups, had an average difference smaller than R, so there was no significant difference.

1.2. Discussion

1.2.1. The Effect of Self-Regulated Learning on Students' Advanced Mathematical Thinking (AMT)

The results showed that self-regulated learning significantly affects students' Advanced Mathematical Thinking (AMT) abilities. AMT refers to higher-order cognitive processes that encompass deep conceptual understanding, reflective reasoning, abstraction, generalization, and the ability to analyze complex mathematical structures. These abilities extend beyond procedural competence, requiring students to engage in sophisticated mathematical reasoning and problem-solving.

Self-regulated learning (SRL) is a crucial factor supporting the development of such higher-order thinking skills. Students with high levels of SRL actively manage their learning processes by setting goals, monitoring progress, regulating strategies, and evaluating performance (Runisah et al., 2020). These self-regulatory behaviors enable deeper engagement with mathematical concepts and facilitate the development of advanced mathematical thinking. The study's findings indicate that students with high levels of SRL demonstrate a better understanding of abstract mathematical concepts and achieve higher AMT performance than those with lower levels.

The importance of SRL in developing mathematical abilities is also supported by previous studies (Maulida et al., 2024; Ilmi et al., 2022), which reported that students with strong SRL skills tend to employ more effective learning strategies and achieve higher levels of complex mathematical performance. Furthermore, students with high SRL are more likely to generate creative and innovative solutions to mathematical problems, reflecting stronger mathematical creativity and reasoning abilities (Runisah et al., 2020). Since creativity, abstraction, and reasoning are essential components of AMT, these findings reinforce the conclusion that SRL plays a significant role in enhancing students' Advanced Mathematical Thinking abilities.

1.2.2. he Effect of Learning Models on Students' Advanced Mathematical Thinking (AMT)

The findings indicate that the M-APOS model produced the highest AMT scores compared with the Knisley and Discovery Learning models. The M-APOS model emphasizes students' mental construction of mathematical knowledge through four stages: Action, Process, Object, and Schema. Through these stages, students progressively develop conceptual understanding by transforming procedural actions into abstract mathematical structures. This process enables students to represent mathematical concepts in multiple forms, including symbols, notations, diagrams, and graphs (Tamalene et al., 2022).

The superior performance of students taught using the M-APOS model may be attributed to its emphasis on abstraction, conceptual construction, and mathematical reasoning, which are essential components of Advanced Mathematical Thinking. During M-APOS learning activities, students are encouraged to construct mathematical concepts independently, solve non-routine problems, and explore alternative solution strategies (Suryana & Seruni, 2022). Previous studies have also shown that the M-APOS model promotes mathematical creativity and supports students in developing proof construction skills and higher-order reasoning abilities (Marsitin, 2017; Ilmi et al., 2022). These characteristics explain why students in the M-APOS group achieved higher AMT scores than those in the other instructional groups.

The Knisley model also demonstrated positive effects on students' AMT abilities, producing higher learning outcomes than the Discovery Learning model. This model emphasizes active student engagement and conceptual development through a sequence of learning stages that encourage students to connect prior knowledge with new mathematical ideas. Learning through the Knisley model enhances conceptual understanding and mathematical representation skills, both of which are important dimensions of Advanced Mathematical Thinking (Fitriani & Nurfauziah, 2020). By encouraging students to understand concepts deeply and apply them in different mathematical contexts, the Knisley model contributes to the development of higher-order mathematical reasoning.

In contrast, although the Discovery Learning model promotes concept acquisition through exploration and investigation, its effectiveness in developing AMT was lower than that of the M-APOS and Knisley models. The findings suggest that Discovery Learning may not provide sufficient cognitive scaffolding for students to develop advanced abstraction and proof-related skills, particularly for those with low levels of self-regulated learning. Students with limited self-regulation may experience difficulties in independently organizing information, monitoring their understanding, and constructing abstract mathematical concepts. Previous research has reported that Discovery Learning becomes more effective when supported by additional instructional guidance or learning media (Pasaribu & Fisher, 2023). Therefore, while Discovery Learning can facilitate conceptual discovery, it may require greater instructional support to effectively foster Advanced Mathematical Thinking.

1.2.3. The Interaction Effect of Self-Regulated Learning and Learning Models on Students' Advanced Mathematical Thinking (AMT)

Self-Regulated Learning is the ability of students to independently manage their learning process by planning, monitoring, controlling, and evaluating their learning activities. Students with high Self-Regulated Learning generally demonstrate greater initiative, perseverance, and responsibility in their studies. Various previous studies have shown that Self-Regulated Learning is associated with higher-order thinking skills, as it enables students to employ more effective cognitive and metacognitive strategies for understanding concepts and solving mathematical problems (Hodiyanto & Firdaus, 2020).

However, the results of this study indicated that self-regulated learning does not significantly affect students' Advanced Mathematical Thinking (AMT) ability ($F = 1.47$; $p = 0.235 > 0.05$). This finding suggests that, in general, students with high SRL do not possess significantly better AMT abilities than those with low Self-Regulated Learning. These results imply that AMT ability is not solely determined by individual student characteristics but is also influenced by the environment and learning processes that support the development of abstract thinking skills, mathematical reasoning, mathematical representation, and mathematical proof.

Nevertheless, the analysis also revealed a significant interaction between learning models and self-regulated learning on students' AMT ability ($F = 3.66$; $p = 0.018 < 0.05$). This finding indicates that the effectiveness of a learning model depends on the students' level of self-regulated learning. In other words, the effect of SRL on AMT ability cannot be understood in isolation from the learning model employed. These results demonstrate that student characteristics and learning strategies interact to influence the development of advanced mathematical thinking skills.

The study also showed that the M-APOS model yielded better results than the Knisley and Discovery Learning models. The advantages of the M-APOS model can be attributed to its stages—Action, Process, Object, and Schema—which provide students with opportunities to construct mathematical concepts gradually and systematically. Through these stages, students not only understand mathematical procedures but are also able to build representations, perform abstractions, develop reasoning, and construct mathematical proofs. This characteristic aligns with the components of Advanced Mathematical Thinking, which include representation, abstraction, mathematical creative thinking, and mathematical proof. These findings are consistent with previous research demonstrating that APOS-based approaches can improve mathematical representation skills, concept understanding, and high-level mathematical thinking skills (Aristika, 2023).

The significant interaction observed indicates that the relationship between self-regulated learning and Advanced Mathematical Thinking (AMT) ability varies across different learning models. This suggests that the effectiveness of a learning model is intrinsically linked to students' self-regulation in their learning process. These findings imply that specific learning models may offer more optimal support for the development of AMT capabilities than others. In this study, the M-APOS model yielded superior results compared to the Knisley and Discovery Learning models. This suggests that the systematic learning structure inherent in M-APOS has the potential to aid students in constructing abstract concepts and fostering advanced mathematical thinking skills.

Conversely, the Knisley and Discovery Learning models prioritize active student involvement in independent exploration and discovery of concepts. While this approach has merits in promoting engagement, its effectiveness can be influenced by students' capacity for self-organized learning. Consequently, the relationship between self-regulated learning and AMT ability becomes more intricate, contingent on the characteristics of the applied learning model. The findings of this study demonstrate that the enhancement of AMT ability relies not only on students' internal factors but also on the congruence between student characteristics and the chosen learning model. Therefore, educators should consider students' levels of self-regulated learning when selecting a teaching strategy.

The M-APOS Model presents an effective alternative for teaching abstract algebra, as it facilitates the development of AMT capabilities through a systematic and structured concept construction process. Theoretically, these results broaden the application of APOS theory within the context of Abstract Algebra by illustrating the significant role of learning models in AMT development, with the influence of self-regulated learning being more evident through its interaction with these models. Practically, this study implies that implementing the M-APOS model can assist educators in optimizing the development of students' advanced mathematical thinking skills. Future research could investigate the interaction between learning models and other factors, such as mathematical disposition, learning motivation, or prior mathematical ability, to achieve a more comprehensive understanding of the influences on Advanced Mathematical Thinking.

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